

Constraints on Operator Ordering from Third Quantization

Yoshiaki Ohkuwa¹, Mir Faizal², Yasuo Ezawa³

¹Division of Mathematical Science,
Department of Social Medicine,
Faculty of Medicine, University of Miyazaki,
Kihara 5200, Kiyotake-cho, Miyazaki, 889-1692, Japan

²Department of Physics and Astronomy,
University of Waterloo, Waterloo,
Ontario N2L 3G1, Canada

³Department of Physics, Ehime university,
2-5 Bunkyo-cho, Matsuyama, 790-8577, Japan

Abstract

In this paper, we analyse the Wheeler-DeWitt equation in the third quantized formalism. We will demonstrate that for certain operator ordering, the early stages of the universe are dominated by quantum fluctuations, and the universe becomes classical at later stages during the cosmic expansion. This is physically expected, if the universe is formed from quantum fluctuations in the third quantized formalism. So, we will argue that this physical requirement can be used to constrain the form of the operator ordering chosen. We will explicitly demonstrate this to be the case for two different cosmological models.

PACS numbers : 04.60.Ds, 04.60.Kz, 98.80.Qc

1 Introduction

The wave function of the universe contains all the physical information about the universe because it describes the quantum state of the universe [1]-[2]. In the no-boundary proposal the wave function of the universe is obtained by summing over all four geometries and field configurations that match a specific field configuration on a spatial section. The wave function can also be obtained as a solution to the Wheeler-DeWitt equation, which can be viewed as the Schrödinger's equation for gravity [3]-[4]. However, just as the single particle Schrödinger's equation cannot be used to analyse a multi-particle system in

the first quantized formalism, the Wheeler-DeWitt equation cannot be used to describe multi-universe system in the second quantized formalism. However, a multi-particle system can be analysed by using second quantization. This observation has motivated the study of third quantization of the Wheeler-DeWitt equation [5]-[10]. In the third quantized formalism, the Wheeler-DeWitt equation is viewed as a classical field equation. Thus, an action is constructed such that the field equations corresponding to that action are the Wheeler-DeWitt equation. This action describes the theory that can be third quantized. The third quantization of this action produces a multi-geometry theory. If these geometries are identified with individual universes, then the third quantization of this action describes a multiverse [11]-[16].

The third quantization has also been used for analysing the virtual black holes [17]. Here the fluctuations of the spacetime at Planck scale cause the formation of virtual black holes. This model of virtual black holes can be used to solve the problem of time. This is because in this model the entropy of the universe keeps increasing due to the interaction of these virtual black holes with matter. The direction of time can then be identified with the increase of the total entropy of the universe. The model for the spacetime foam can also be used to explain the end stage of the evaporation of real black holes [18]. In this model, real black holes evaporate down to Planck size and then disappear in the sea of virtual black holes. It may be noted that the third quantization has also been used to address the cosmological constant problem using the idea of baby universes [19]. In this model, the creation or annihilation of baby of universe in the third quantized formalism is similar to the creation or annihilation of a particle in the second quantized formalism. The propagator of the theory corresponded to a wormhole, and the third quantized version of the momentum conservation is represented by the conservation of the axion charge.

It is possible for the universe to form from quantum fluctuations in the third quantized formalism [20]. Thus, it is expected that the early stages in the evolution of the universe would be dominated by quantum fluctuations. It is known that the geometry of spacetime is described by a classical spacetime at later stages, so it is expected that the quantum fluctuations will get minimized at later stages of the cosmic expansion. The uncertainty for a model of third quantized universes has been studied, and it was observed that the fluctuations in the third quantized formalism decrease very rapidly during the course of cosmic expansion [21]-[22]. This uncertainty has also been studied for third quantized Brans-Dicke theories [23]. In this analysis, the distribution function for the universes has been obtained. The uncertainty principle has also been discussed in the context of the third quantization of $f(R)$ gravity theories [24]-[25]. The distribution function for the universes in the third quantization Kaluza-Klein theories have also been obtained [26]. In this analysis, it was demonstrated that the compactification of geometries is consistent with third quantization. The third quantization has been used to analyse the quantum transitions from the string perturbative vacuum to cosmological configurations which is characterized by isotropic contraction and decreasing dilaton [27]. It was observed that such transitions could be represented by the production of pairs of universes from the vacuum state. All this analysis was done for specific choice of operator ordering. However, it is known that operator ordering can have direct physical consequences [28]-[29]. So, this motivates us to study the effect of operator ordering on the third quantization, and this is what will be

done in this paper. We would like to point out that the main aim of this paper is to use the physical requirements on cosmological models to restrict the form of operator ordering used. This is because there is no mathematical way to prefer one choice of factor ordering from another.

In this paper, we study the effect of operator ordering in the third quantized formalism. We will use the third quantized formalism for analysing the effect of uncertainty relation on the structure of spacetime during cosmic expansion. We will also discuss the operator ordering for this theory. We will observe that for a specific choice of operator ordering quantum fluctuations dominate at the early stage of the universe and the spacetime becomes classical at the later stages. This is physically expected, if the universe is formed from quantum fluctuations in the third quantized formalism. The remaining of the paper is organized as follows. In section 2, we will analyse the third quantization of general relativity with a cosmological constant. In section 3, we will then study the uncertainty relation for this model, and in section 4, we will analyse the operator ordering for this model. Then in section 5, we will apply this formalism for another minisuperspace model. Finally, we will summarize our results in section 6. We will also suggest some possible extension of this works in this last section.

2 Einstein Gravity

The universe is expected to form from the quantum fluctuations which could be described by the third quantized theory [20]. So, we would need to analyse the Wheeler-DeWitt equation in the third quantized formalism. So, in this section, we will analyse the third quantization of general relativity with a cosmological constant, since we are interested in operator ordering problem which will be discussed precisely in section 4. The action for general relativity with a cosmological constant term is written as

$$S = \int d^4x \sqrt{-g} \frac{1}{16\pi G} (R - 2\Lambda) . \quad (2.1)$$

Motivated by the cosmological observation [30]-[31], we take the case of a flat Friedmann-Lemaitre-Robertson-Walker metric,

$$ds^2 = -dt^2 + a^2(t) \sum_{k=1}^3 (dx^k)^2 . \quad (2.2)$$

The action for this metric can be written as

$$S = \int dt L , \quad L = \frac{1}{16\pi G} (-6a\dot{a}^2 - 2\Lambda a^3) , \quad (2.3)$$

where we neglected an irrelevant constant. Here a is the scalar factor of the universe. The constant we neglected does not effect the dynamics of the equations of motion from this Lagrangian, and hence can be neglected.

Now we define unit such that $8\pi G = 1$, and obtain the Hamiltonian constraint as

$$\mathcal{H} = -\frac{1}{12a} p_a^2 + \Lambda a^3 \approx 0 , \quad (2.4)$$

where p_a is the canonical momentum of a . Now using the standard representation for the canonical momentum,

$$p_a \rightarrow -i \frac{d}{da} , \quad (2.5)$$

we obtain the Wheeler-DeWitt equation

$$\left[\frac{1}{a^{p_o}} \frac{d}{da} a^{p_o} \frac{d}{da} + 12\Lambda a^4 \right] \psi(a) = 0 .$$

We can also write it as

$$\left[\frac{d^2}{da^2} + \frac{p_o}{a} \frac{d}{da} + 12\Lambda a^4 \right] \psi(a) = 0 . \quad (2.6)$$

Here p_o is the operator ordering parameter. The dependence of wave function of the universe on the cosmological constant has already been studied [1]. In this paper, we will analyse the effect of operator ordering on the quantum fluctuations in quantum cosmology.

We consider a as time. The Lagrangian for the third quantization which yields Eq. (2.6) is

$$\mathcal{L}_{3Q} = \frac{1}{2} \left[a^{p_o} \left(\frac{d\psi(a)}{da} \right)^2 - 12\Lambda a^{p_o+4} \psi(a)^2 \right] . \quad (2.7)$$

Note that if we define

$$S_{3Q} = \int da \mathcal{L}_{3Q} , \quad (2.8)$$

then we obtain Eq. (2.6) from $\delta S_{3Q} = 0$.

The canonical momentum for $\psi(a)$ is defined as

$$\pi(a) = \frac{\partial \mathcal{L}_{3Q}}{\partial \left(\frac{d\psi(a)}{da} \right)} = a^{p_o} \frac{d\psi(a)}{da} . \quad (2.9)$$

The Hamiltonian for the third quantization reads

$$\begin{aligned} \mathcal{H}_{3Q} &= \pi(a) \frac{d\psi(a)}{da} - \mathcal{L}_{3Q} , \\ &= \frac{1}{2} \left[\frac{1}{a^{p_o}} \pi(a)^2 + 12\Lambda a^{p_o+4} \psi(a)^2 \right] . \end{aligned} \quad (2.10)$$

Now we third quantize this theory by imposing the equal time commutation relation as

$$[\hat{\psi}(a), \hat{\pi}(a)] = i , \quad (2.11)$$

where hat represents an operator. Taking the Schrödinger picture, we have the time-independent c -number ψ for the operator $\hat{\psi}(a)$, so we can replace the operators as

$$\hat{\psi}(a) \rightarrow \psi , \quad \hat{\pi}(a) \rightarrow -i \frac{\partial}{\partial \psi} . \quad (2.12)$$

Then we obtain the Schrödinger equation

$$i \frac{\partial \Psi(a, \psi)}{\partial a} = \hat{\mathcal{H}}_{3Q} \Psi(a, \psi) , \quad (2.13)$$

$$\hat{\mathcal{H}}_{3Q} = \frac{1}{2} \left[-\frac{1}{a^{p_o}} \frac{\partial^2}{\partial \psi^2} + 12\Lambda a^{p_o+4} \psi^2 \right] ,$$

where $\Psi(a, \psi)$ is the third quantized wave function of universes. This equation is the solution to a third quantized Schrödinger equation, and hence it describes a multi-universe state, just as the solution to a second quantized Schrödinger equation would describe a multi-particle state.

3 Uncertainty Relation

It may be noted that the universe is expected to be formed from quantum fluctuations in the third quantized formalism [20]. Thus, we expect that, at the beginning, the universe will be dominated by quantum fluctuations. However, it is known that the universe is described by a classical geometry at later stages. Thus, we expect that in the third quantized formalism, the early stages of the universe should be dominated by quantum fluctuations, and the later times the universe should be described by a classical geometry. By classical geometry we mean that the quantum fluctuations of the geometry of the universe should get minimized. We can also make the definition of early and later times for the universe more precise by defining the early times by the limit $a \rightarrow 0$, and the later times by the limit $a \rightarrow \infty$.

Now we will analyse the uncertainty relation for the universe to analyse the behavior of quantum fluctuations at different stages of the cosmic expansion. In order to do that, we assume that the solution to Eq. (2.13) has the Gaussian form

$$\Psi(a, \psi) = C \exp \left\{ -\frac{1}{2} A(a) [\psi - \eta(a)]^2 + i B(a) [\psi - \eta(a)] \right\} , \quad (3.1)$$

where C is a constant, $A(a) = D(a) + iI(a)$, and $A(a), B(a), \eta(a)$ should be determined from Eq. (2.13). The inner product of two third quantized wave functions Ψ_1 and Ψ_2 can be defined as

$$\langle \Psi_1, \Psi_2 \rangle = \int d\psi \Psi_1^*(a, \psi) \Psi_2(a, \psi). \quad (3.2)$$

Let us calculate Heisenberg's uncertainty relation. The dispersion of ψ is defined as

$$(\Delta\psi)^2 \equiv \langle \psi^2 \rangle - \langle \psi \rangle^2 , \quad \langle \psi^2 \rangle = \frac{\langle \Psi, \psi^2 \Psi \rangle}{\langle \Psi, \Psi \rangle} . \quad (3.3)$$

Using Eqs. (3.1), (3.2) and (3.3), we have [24]-[25]

$$\langle \psi^2 \rangle = \frac{1}{2D(a)} + \eta^2(a) , \quad \langle \psi \rangle = \eta(a) , \quad \text{and} \quad (\Delta\psi)^2 = \frac{1}{2D(a)} . \quad (3.4)$$

The dispersion of π is defined as

$$(\Delta\pi)^2 \equiv \langle \pi^2 \rangle - \langle \pi \rangle^2 , \quad \langle \pi^2 \rangle = \frac{\langle \Psi, \pi^2 \Psi \rangle}{\langle \Psi, \Psi \rangle} . \quad (3.5)$$

Then we obtain

$$\langle \pi^2 \rangle = \frac{D(a)}{2} + \frac{I^2(a)}{2D(a)} + B^2(a), \quad \langle \pi \rangle = B(a), \quad (3.6)$$

and

$$(\Delta\pi)^2 = \frac{D(a)}{2} + \frac{I^2(a)}{2D(a)}.$$

Therefore the uncertainty can be written as

$$(\Delta\psi)^2(\Delta\pi)^2 = \frac{1}{4} \left(1 + \frac{I^2(a)}{D^2(a)} \right). \quad (3.7)$$

Substituting the assumption (3.1) to Eq. (2.13), we obtain the equation for $A(a)$ as

$$-\frac{i}{2} \frac{dA(a)}{da} = -\frac{1}{2a^{p_o}} A(a)^2 + 6\Lambda a^{p_o+4}. \quad (3.8)$$

(We obtain three equations for $A(a), B(a), \eta(a)$ by comparing the order of ψ in Eq. (2.13), but Eq. (3.8) is enough for the discussion of the Heisenberg uncertainty relation.) Defining

$$\sigma \equiv a^{1-p_o}, \quad (3.9)$$

we obtain

$$-i \frac{1-p_o}{2} \frac{dA(\sigma)}{d\sigma} + \frac{A(\sigma)^2}{2} - 6\Lambda \sigma^{\frac{2p_o+4}{1-p_o}} = 0. \quad (3.10)$$

Here we have assumed $p_o \neq 1$. Let us define a function $u(\sigma)$ by the equation,

$$A(\sigma) = -i(1-p_o) \frac{d \ln u(\sigma)}{d\sigma}. \quad (3.11)$$

Then we have

$$\frac{d^2 u(\sigma)}{d\sigma^2} + \frac{12\Lambda}{(1-p_o)^2} \sigma^{\frac{2p_o+4}{1-p_o}} u(\sigma) = 0. \quad (3.12)$$

This equation can be solved using a Bessel function as

$$u(\sigma) = \sigma^{\frac{1}{2}} \mathcal{B}_{\frac{1-p_o}{6}} \left(2\sqrt{\frac{\Lambda}{3}} \sigma^{\frac{3}{1-p_o}} \right), \quad (3.13)$$

where \mathcal{B} is a Bessel function which satisfies [32]

$$\frac{d^2 \mathcal{B}_{\frac{1-p_o}{6}}(z)}{dz^2} + \frac{1}{z} \frac{d\mathcal{B}_{\frac{1-p_o}{6}}(z)}{dz} + \left(1 - \frac{\left(\frac{1-p_o}{6}\right)^2}{z^2} \right) \mathcal{B}_{\frac{1-p_o}{6}}(z) = 0. \quad (3.14)$$

Therefore we obtain the general solution to eq. (3.12) as

$$u(\sigma) = c_J \sigma^{\frac{1}{2}} J_{\frac{1-p_o}{6}} \left(2\sqrt{\frac{\Lambda}{3}} \sigma^{\frac{3}{1-p_o}} \right) + c_Y \sigma^{\frac{1}{2}} Y_{\frac{1-p_o}{6}} \left(2\sqrt{\frac{\Lambda}{3}} \sigma^{\frac{3}{1-p_o}} \right), \quad (3.15)$$

where c_J and c_Y are arbitrary constants and $J_{\frac{1-p_o}{6}}$ and $Y_{\frac{1-p_o}{6}}$ are Bessel functions.

Now if we define

$$z \equiv 2\sqrt{\frac{\Lambda}{3}}\sigma^{\frac{3}{1-p_o}} = 2\sqrt{\frac{\Lambda}{3}}a^3. \quad (3.16)$$

So, Eq. (3.15) can be written as

$$u(z) = \left(\frac{z}{2\sqrt{\frac{\Lambda}{3}}}\right)^{\frac{1-p_o}{6}} [c_J J_{\frac{1-p_o}{6}}(z) + c_Y Y_{\frac{1-p_o}{6}}(z)]. \quad (3.17)$$

Now from Eqs. (3.11), (3.16), we obtain

$$\begin{aligned} A(z) &= -i(1-p_o)\frac{dz}{d\sigma}\frac{d \ln u(z)}{dz} \\ &= -i6\sqrt{\frac{\Lambda}{3}}\left(\frac{z}{2\sqrt{\frac{\Lambda}{3}}}\right)^{\frac{p_o+2}{3}} \frac{c_J J_{\frac{-5-p_o}{6}}(z) + c_Y Y_{\frac{-5-p_o}{6}}(z)}{c_J J_{\frac{1-p_o}{6}}(z) + c_Y Y_{\frac{1-p_o}{6}}(z)}, \end{aligned} \quad (3.18)$$

where we have used [32]

$$\frac{d\mathcal{B}_{\frac{1-p_o}{6}}(z)}{dz} = \mathcal{B}_{\frac{-5-p_o}{6}}(z) - \frac{1-p_o}{6z}\mathcal{B}_{\frac{1-p_o}{6}}(z). \quad (3.19)$$

As $A(z) = D(z) + iI(z)$, we find (note that $c_J c_Y^* - c_J^* c_Y$ is a pure imaginary number)

$$D(z) = \frac{i6\sqrt{\frac{\Lambda}{3}}\left(\frac{z}{2\sqrt{\frac{\Lambda}{3}}}\right)^{\frac{p_o+2}{3}}}{\pi z |c_J J_{\frac{1-p_o}{6}}(z) + c_Y Y_{\frac{1-p_o}{6}}(z)|^2} (c_J c_Y^* - c_J^* c_Y). \quad (3.20)$$

Here we have used [32]

$$J_{\frac{1-p_o}{6}}(z)Y_{\frac{-5-p_o}{6}}(z) - J_{\frac{-5-p_o}{6}}(z)Y_{\frac{1-p_o}{6}}(z) = \frac{2}{\pi z}, \quad (3.21)$$

and

$$\begin{aligned} I(z) &= -\frac{3\sqrt{\frac{\Lambda}{3}}\left(\frac{z}{2\sqrt{\frac{\Lambda}{3}}}\right)^{\frac{p_o+2}{3}}}{|c_J J_{\frac{1-p_o}{6}}(z) + c_Y Y_{\frac{1-p_o}{6}}(z)|^2} \\ &\times \left[2|c_J|^2 J_{\frac{-5-p_o}{6}}(z)J_{\frac{1-p_o}{6}}(z) + 2|c_Y|^2 Y_{\frac{-5-p_o}{6}}(z)Y_{\frac{1-p_o}{6}}(z) \right. \\ &\quad \left. + (c_J c_Y^* + c_J^* c_Y) \left(J_{\frac{-5-p_o}{6}}(z)Y_{\frac{1-p_o}{6}}(z) \right. \right. \\ &\quad \left. \left. + J_{\frac{1-p_o}{6}}(z)Y_{\frac{-5-p_o}{6}}(z) \right) \right]. \end{aligned} \quad (3.22)$$

Therefore if we assume $c_J c_Y^* - c_J^* c_Y \neq 0$ (note that in this case both of c_J, c_Y

are nonzero), we obtain

$$\begin{aligned}
\frac{I(z)^2}{D(z)^2} = & -\frac{\pi^2 z^2}{4(c_J c_Y^* - c_J^* c_Y)^2} \\
& \times \left[2|c_J|^2 J_{\frac{-5-p_o}{6}}(z) J_{\frac{1-p_o}{6}}(z) + 2|c_Y|^2 Y_{\frac{-5-p_o}{6}}(z) Y_{\frac{1-p_o}{6}}(z) \right. \\
& + (c_J c_Y^* + c_J^* c_Y) \left(J_{\frac{-5-p_o}{6}}(z) Y_{\frac{1-p_o}{6}}(z) \right. \\
& \left. \left. + J_{\frac{1-p_o}{6}}(z) Y_{\frac{-5-p_o}{6}}(z) \right) \right]^2 . \tag{3.23}
\end{aligned}$$

Substituting Eq. (3.23) to Eq. (3.7), we obtain the Heisenberg uncertainty relation. This uncertainty relation can be used to analyse the behavior of the geometry during the cosmic expansion.

4 Operator Ordering

In this section, we will discuss the operator ordering for the third quantization of the model studied in the previous sections. We will estimate the order of the uncertainty relation both at the late and early times during the cosmic expansion.

The late times for the cosmic expansion can be defined as $a \rightarrow \infty$ i. e., $z \rightarrow \infty$ from Eq. (3.16). Now we can write [32]

$$J_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right), \quad Y_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right), \tag{4.1}$$

where $\nu = \frac{-5-p_o}{6}$ and $\frac{1-p_o}{6}$, so we obtain from Eq. (3.23)

$$\begin{aligned}
\frac{I(z)^2}{D(z)^2} \sim & -\frac{1}{(c_J c_Y^* - c_J^* c_Y)^2} \\
& \times \left[2|c_J|^2 \cos\left(z + \frac{p_o+2}{12}\pi\right) \cos\left(z + \frac{p_o-4}{12}\pi\right) \right. \\
& + 2|c_Y|^2 \sin\left(z + \frac{p_o+2}{12}\pi\right) \sin\left(z + \frac{p_o-4}{12}\pi\right) \\
& \left. + (c_J c_Y^* + c_J^* c_Y) \sin\left(2z + \frac{p_o-1}{6}\pi\right) \right]^2 \\
& \sim O(1) . \tag{4.2}
\end{aligned}$$

This along with Eq. (3.7) indicate that at late times, i. e., in the limit $a \rightarrow \infty$, the spacetime can become classical in the sense that the quantum fluctuations become minimum.

On the other hand at early times namely when $a \rightarrow 0$ i.e., $z \rightarrow 0$ from Eq. (3.16), we must divide the cases by the value of the operator ordering parameter p_o .

For example, when we choose as the usual case

$$p_o = -1, \quad (4.3)$$

we have for early times [32]

$$\begin{aligned} J_{\frac{1}{3}}(z) &\sim \frac{1}{\Gamma\left(\frac{4}{3}\right)} \left(\frac{z}{2}\right)^{\frac{1}{3}}, & J_{-\frac{2}{3}}(z) &\sim \frac{1}{\Gamma\left(\frac{1}{3}\right)} \left(\frac{z}{2}\right)^{-\frac{2}{3}}, \\ Y_{\frac{1}{3}}(z) &\sim -\frac{1}{\pi} \Gamma\left(\frac{1}{3}\right) \left(\frac{z}{2}\right)^{-\frac{1}{3}}, & Y_{-\frac{2}{3}}(z) &\sim \frac{1}{\sqrt{3}\Gamma\left(\frac{1}{3}\right)} \left(\frac{z}{2}\right)^{-\frac{2}{3}}. \end{aligned} \quad (4.4)$$

So, we can use Eq. (3.23) to obtain

$$\begin{aligned} \frac{I(z)^2}{D(z)^2} &\sim -\frac{1}{(c_J c_Y^* - c_J^* c_Y)^2} \left[\frac{2}{\sqrt{3}} |c_Y|^2 + (c_J c_Y^* + c_J^* c_Y) \right]^2 \\ &\sim O(1). \end{aligned} \quad (4.5)$$

This along with Eq. (3.7) indicate that at early times, i. e., in the limit $a \rightarrow 0$, the spacetime can again become classical. Thus, this value of the operator ordering does not seem to be physically plausible as the spacetime cannot become classical at early times, and we do expect that at early times the quantum fluctuations should dominate.

On other hand, when we choose

$$p_o = -5, \quad (4.6)$$

we have for early times [32]

$$\begin{aligned} J_0(z) &\sim 1 - \frac{z^2}{4}, & J_1(z) &\sim \frac{z}{2}, \\ Y_0(z) &\sim \frac{2}{\pi} \ln z, & Y_1(z) &\sim -\frac{2}{\pi z}. \end{aligned} \quad (4.7)$$

So, we can write

$$\frac{I^2(z)}{D^2(z)} \sim -\frac{16|c_Y|^4}{\pi^2(c_J c_Y^* - c_J^* c_Y)^2} (\ln z)^2 \sim \infty. \quad (4.8)$$

This along with Eq.(3.7) indicate that the fluctuation of the universe field becomes very large at early times, i. e., in the limit $a \rightarrow 0$ in the third quantized formalism. Therefore the quantum fluctuations dominate the structure of spacetime for the small values of the scale factor of the universe. Thus, it seems to be the physically plausible result as we do expect the quantum fluctuations to dominate the universe at the very early stages. It may be noted that a similar result has been discussed in the context of $f(R)$ gravity [24]-[25]. Here we have seen that the physics of the system might depend critically on operator ordering. This can be used to put a constraint on the operator ordering parameter, allowing only those values which give physically expected results.

5 Another Model for the Universe

It is important to check if the observations of the previous sections were a specific feature of the particular minisuperspace model we analysed. Thus, it is important to perform a similar analysis for a different model for the universe. So, in this section, we analyse another model for the universe. We will study the third quantization of a closed universe which is filled with constant vacuum energy density and radiation. The Wheeler-DeWitt equation for this model can be written as [33]

$$\left[\frac{d^2}{da^2} + \frac{p_o}{a} \frac{d}{da} - k_2 a^2 + k_4 \rho_v a^4 + k_0 \epsilon \right] \psi(a) = 0 . \quad (5.1)$$

Here a is again the scale factor of the universe, p_o is the operator ordering parameter, and [33]

$$k_2 = \frac{9\pi^2}{4G^2\hbar^2} , \quad k_4 = \frac{6\pi^3}{G\hbar^2} , \quad k_0 = \frac{6\pi^3}{G\hbar^2} . \quad (5.2)$$

The behavior of the wave function of the universe, and its dependence on the vacuum energy and radiation has been studied [34]. We have denoted the vacuum energy density by ρ_v , and the radiation by ϵ . Now repeating the analysis performed for the previous model, we obtain the Schrödinger equation for this model,

$$i \frac{\partial \Psi(a, \psi)}{\partial a} = \hat{\mathcal{H}}_{3Q} \Psi(a, \psi) , \quad (5.3)$$

$$\hat{\mathcal{H}}_{3Q} = \frac{1}{2} \left[-\frac{1}{a^{p_o}} \frac{\partial^2}{\partial \psi^2} + a^{p_o} (-k_2 a^2 + k_4 \rho_v a^4 + k_0 \epsilon) \psi^2 \right] ,$$

where $\Psi(a, \psi)$ is the third quantized wave function.

If we assume that the solution to Eq. (5.3) is the same Gaussian form of Eq. (3.1), we obtain the equation for $A(a)$ as

$$-\frac{i}{2} \frac{dA(a)}{da} = -\frac{1}{2a^{p_o}} A(a)^2 + \frac{a^{p_o}}{2} (-k_2 a^2 + k_4 \rho_v a^4 + k_0 \epsilon) . \quad (5.4)$$

Using the same equations as Eqs. (3.9) and (3.11), we have

$$\frac{d^2 u(\sigma)}{d\sigma^2} + \frac{1}{(1-p_o)^2} \left(-k_2 \sigma^{\frac{2p_o+2}{1-p_o}} + k_4 \rho_v \sigma^{\frac{2p_o+4}{1-p_o}} + k_0 \epsilon \sigma^{\frac{2p_o}{1-p_o}} \right) u(\sigma) = 0 . \quad (5.5)$$

Though this equation is too complicated to solve exactly, we need only the limiting cases for the late times $a \rightarrow \infty$ and the early times $a \rightarrow 0$, so we will solve it in these limiting cases.

At the late times we obtain from Eqs. (5.4) and (5.5)

$$\frac{d^2 u(\sigma)}{d\sigma^2} + \frac{1}{(1-p_o)^2} k_4 \rho_v \sigma^{\frac{2p_o+4}{1-p_o}} u(\sigma) = 0 . \quad (5.6)$$

Since this equation is essentially the same one as Eq. (3.12), we obtain the same result that is at late times, i.e., in the limit $a \rightarrow \infty$, the spacetime becomes classical in the sense that the quantum fluctuations get minimized.

At early times, we obtain from Eqs. (5.4) and (5.5)

$$\frac{d^2 u(\sigma)}{d\sigma^2} + \frac{1}{(1-p_o)^2} k_0 \epsilon \sigma^{\frac{2p_o}{1-p_o}} u(\sigma) = 0. \quad (5.7)$$

The solution for this is given by [32],

$$u(\sigma) = c_J \sigma^{\frac{1}{2}} J_{\frac{1-p_o}{2}} \left(\sqrt{k_0 \epsilon} \sigma^{\frac{1}{1-p_o}} \right) + c_Y \sigma^{\frac{1}{2}} Y_{\frac{1-p_o}{2}} \left(\sqrt{k_0 \epsilon} \sigma^{\frac{1}{1-p_o}} \right), \quad (5.8)$$

where c_J and c_Y are arbitrary constants. If we define

$$z \equiv \sqrt{k_0 \epsilon} \sigma^{\frac{1}{1-p_o}} = \sqrt{k_0 \epsilon} a, \quad (5.9)$$

then we have

$$u(z) = \left(\frac{z}{\sqrt{k_0 \epsilon}} \right)^{\frac{1-p_o}{2}} [c_J J_{\frac{1-p_o}{2}}(z) + c_Y Y_{\frac{1-p_o}{2}}(z)]. \quad (5.10)$$

So, from Eqs. (3.11), (5.9), (5.10), we obtain [32]

$$A(z) = -i \sqrt{k_0 \epsilon} \left(\frac{z}{\sqrt{k_0 \epsilon}} \right)^{p_o} \frac{c_J J_{\frac{1-p_o}{2}}(z) + c_Y Y_{\frac{1-p_o}{2}}(z)}{c_J J_{\frac{1-p_o}{2}}(z) + c_Y Y_{\frac{1-p_o}{2}}(z)}. \quad (5.11)$$

Now repeating the analysis of the previous section, we obtain [32]

$$\begin{aligned} \frac{I(z)^2}{D(z)^2} = & -\frac{\pi^2 z^2}{4(c_J c_Y^* - c_J^* c_Y)^2} \\ & \times \left[2|c_J|^2 J_{\frac{1-p_o}{2}}(z) J_{\frac{1-p_o}{2}}(z) + 2|c_Y|^2 Y_{\frac{1-p_o}{2}}(z) Y_{\frac{1-p_o}{2}}(z) \right. \\ & + (c_J c_Y^* + c_J^* c_Y) \left(J_{\frac{1-p_o}{2}}(z) Y_{\frac{1-p_o}{2}}(z) \right. \\ & \left. \left. + J_{\frac{1-p_o}{2}}(z) Y_{\frac{1-p_o}{2}}(z) \right) \right]^2. \end{aligned} \quad (5.12)$$

If we choose as the usual case

$$p_o = -1, \quad (5.13)$$

we have the same relation as (4.8). So, this along with Eq.(3.7) indicate that the fluctuation of the third quantized universe field becomes large at early times namely $a \rightarrow 0$. Therefore the quantum effects dominate for the small values of the scale factor of the universe.

However if we choose other operator ordering parameter for example

$$p_o = 0, \quad (5.14)$$

at early times, we have [32]

$$\begin{aligned} J_{\frac{1}{2}}(z) & \sim \frac{2}{\sqrt{\pi}} \left(\frac{z}{2} \right)^{\frac{1}{2}}, & J_{-\frac{1}{2}}(z) & \sim \frac{1}{\sqrt{\pi}} \left(\frac{z}{2} \right)^{-\frac{1}{2}}, \\ Y_{\frac{1}{2}}(z) & \sim -\frac{1}{\sqrt{\pi}} \left(\frac{z}{2} \right)^{-\frac{1}{2}}, & Y_{-\frac{1}{2}}(z) & \sim \frac{2}{\sqrt{\pi}} \left(\frac{z}{2} \right)^{\frac{1}{2}}. \end{aligned} \quad (5.15)$$

So, we can obtain from Eq. (5.12)

$$\frac{I(z)^2}{D(z)^2} \sim -\frac{(c_J c_Y^* + c_J^* c_Y)^2}{(c_J c_Y^* - c_J^* c_Y)^2} \sim O(1). \quad (5.16)$$

This along with Eq. (3.7) indicate that at early times, i. e., in the limit $a \rightarrow 0$, the spacetime becomes classical. This does not seem like a physical result. Thus, again we have observed that the physics of this system depends on operator ordering, and this can be used to constrain the form of the operator ordering. So, we have demonstrated for two different models that the physical requirement that quantum fluctuations dominate the early stages of the universe, and spacetime becomes classical at later stages in the cosmic expansion, holds for certain values of operator ordering.

For the first model, we observe that $p_0 = -1$ is not physically plausible, and $p_0 = -5$ is physically acceptable value for the operator ordering. On the other hand, we observe that for the second model, $p_0 = -1$ is physically acceptable, and $p_0 = 0$ is not physically acceptable value of the operator ordering. Thus, it seems that the exact value of the operator ordering chosen depends on the physical content of the universe. Hence, there seems to be no universal value for the operator ordering that can be chosen for all the physical problems, and rather a different value of the operator ordering has to be chosen for different models of quantum cosmology. However, the fact that some values of the operator ordering will give physically plausible results, and other values will give physically acceptable results, seems to be a general feature of all different models in quantum cosmology. This is rather a surprising results, as one would have expected that any physical dependence on operator ordering should be a universal feature of all the models of quantum cosmology. However, in this paper, we have demonstrated that the physically acceptable value for the operator ordering to be model dependent.

It may be noted that for both the cosmological models, we found that the quantum fluctuations depended on the exact value of the operator ordering chose. In the first model the universe was filled with a cosmological constant, and in the second model there was an important contribution coming from the radiation. So, the exact form of the Wheeler-DeWitt equation was different for both these models, and hence the wave function of the universe had a different form for both these models. However, it was interesting to note that in both these models the quantum fluctuations in the early universe depended critically on the operator ordering chosen. There is no mathematical way to prefer a specific form of operator ordering in the Wheeler-DeWitt equation. However, in this paper, we have demonstrated that operator ordering can have non-trivial physical consequences. So, it is possible to use the physical requirement on the form of the wave function for the universe to prefer a specific form of the operator ordering. We do require the geometry of the universe to be dominated by quantum fluctuations at the very early stage, and this quantum state of universe is expected to give rise to a classical geometry at later stages. It has been demonstrated in this paper, that this requirement can be used to prefer a choice of operator ordering in the Wheeler-deWitt equation. However, the exact value of the operator ordering will also depend on the details of the cosmological model being studied.

6 Conclusion

In this paper, we studied the operator ordering in the third quantized formalism. This was done by analysing the the Wheeler-DeWitt equation in the third quantized formalism. Thus, an action was constructed such that the Wheeler-DeWitt equation was obtained as its field equation. This action was then third quantized. We studied the uncertainty relation for this third quantized theory of gravity. We also discussed the operator order for this theory. We observed that the physical requirements are satisfied for certain values of operator ordering. It is possible for the universe to be formed from quantum fluctuations in this third quantized formalism [20]. So, it is expected that the early stages of the universe will be dominated by quantum fluctuations. Furthermore, it is known that the geometry of the universe has to become classical at later stages of the cosmic expansion. In this paper, we demonstrate that for certain values of operator ordering parameter, the quantum fluctuations do dominate the early state of the universe, and the spacetime geometry does becomes classical, at the later stages of the cosmic evolution. Hence, we can use this to put constraints on the form of operator ordering chosen. It may be noted that we have demonstrate this to be the case for two different types of model. However, the exact value of the operator ordering chosen depends on the details of the model being studied. We would like to point out that factor ordering occur as there is an ambiguity in defining two quantum operators at the same point, and this is similar to the occurrence of an anomaly in quantum field theory. In fact, it has been demonstrated that the factor ordering in the Wheeler-DeWitt equation can effect the classical Friedman equation [33].

It may be noted that the third quantization has also been studied in the context of group field cosmology [35]-[38]. The FFBRST for the group field cosmology has also been studied [39]-[40]. As recently lot of progress has been made on FFBRST [41]-[46], it would be interesting to analyse the implications of these results on group field cosmology. It will be also interesting to analyse the uncertainty in the context of third quantized Horava-Lifshitz gravity [47]-[48]. This theory is based on the idea of modifying the scaling of space and time in such a way that the infrared limit of this theory coincides with general relativity. Thus, Horava-Lifshitz gravity is viewed as the ultraviolet completion of general relativity. This theory uses the concept of Lifshitz scaling from solid state physics, it is generally called Horava-Lifshitz theory $t \rightarrow b^z t$, $\mathbf{x} \rightarrow b\mathbf{x}$, where z is called the dynamical critical exponent z , and we can assume that $z = 3$ [47]-[48]. The Wheeler-DeWitt equation for the Horava-Lifshitz gravity has been studied [50]-[51]. The Wheeler-DeWitt equation for the Horava-Lifshitz gravity has also been used to study the cosmological constant problem [52]. The third quantization of the Horava-Lifshitz gravity has also been studied [11]. This was done by associating the eigenvalues of the Wheeler-DeWitt equation with the cosmological constant. It will be interesting to study the uncertainty for the third quantized Horava-Lifshitz theory of gravity.

The Horava-Lifshitz theory of gravity is closely related to the gravity's rainbow [53]. The deformation of black holes has been studied using gravity's rainbow [54]-[59]. The Wheeler-DeWitt equation for the gravity's rainbow has already been studied using the second quantized formalism [60]-[61]. It will be interesting to perform the third quantization of the Wheeler-DeWitt equation for gravity's rainbow. This is because this third quantized Wheeler-DeWitt

equation for gravity's rainbow can be used for analysing different aspects of virtual black holes. It may be noted that a deformation of the kinetic part of the Wheeler-DeWitt equation has been recently studied [62]-[65]. This deformation of the Wheeler-DeWitt equation is based on generalized uncertainty principle [66]-[67]. The third quantization of this deformed Wheeler-DeWitt equation has also been discussed [68]. It will be interesting to study the uncertainty in the context of the third quantization of a deformed Wheeler-DeWitt equation. We can also discuss the operator order for deformed Wheeler-DeWitt equation. It may be noted that it is possible to study this deformation of the Wheeler-DeWitt equation for the various modified theories of gravity like the $f(R)$ gravity and the Horava-Lifshitz gravity. It will be interesting to study the effect of operator ordering for these modifications to the Wheeler-DeWitt equation.

References

- [1] J. B. Hartle and S. W. Hawking, Phys. Rev. D 28, 2960 (1983)
- [2] S. W. Hawking, T. Hertog and H. Reall, Phys. Rev. D 62, 043501 (2000)
- [3] B. S. DeWitt, Phys. Rev. 160, 1113 (1967)
- [4] J. A. Wheeler, Ann. Phys. 2, 604 (1957)
- [5] N. Caderni and M. Martellini, Int. J. Theor. Phys. 23, 233 (1984)
- [6] M. McGuigan, Phys. Rev. D 38, 3031 (1988)
- [7] M. Faizal, Class. Quant. Grav. 29, 215009 (2012)
- [8] Y. Peleg, Class. Quant. Grav. 8, 827 (1991)
- [9] M. Faizal, Grav. Cosmol. 20, 132 (2014)
- [10] Y. Peleg, Mod. Phys. Lett. A 8, 1849 (1993)
- [11] M. Faizal, Mod. Phys. Lett. A 27, 1250007 (2012)
- [12] P. F. G. Diaz and S. R. Perez, Int. J. Mod. Phys. D 17, 1213 (2008)
- [13] S. R. Perez, Y. Hassouni and P. F. G. Diaz, Phys. Lett. B 683, 1 (2010)
- [14] S. R. Perez and P. F. G. Diaz, Phys. Rev. D 81, 083529 (2010)
- [15] I. Garay and S. R. Perez, Int. J. Mod. Phys. D 23, 1450043 (2014)
- [16] M. Faizal, Comm. Theor. Phys. 62, 697 (2014)
- [17] M. Faizal, JETP. 114, 400 (2012)
- [18] S. W. Hawking, Phys. Rev. D 53, 3099 (1996)
- [19] S. B. Giddings and A. Strominger, Nucl. Phys. B 321, 481 (1989)
- [20] M. McGuigan, Phys. Rev. D 39, 2229 (1989)
- [21] T. Horiguchi, Phys Rev. D 48, 5764 (1993)

- [22] H. J. Pohle, Phys. Lett. B 261, 257 (1991)
- [23] L. O. Pimentel and C. Mora, Phys. Lett. A 280, 191 (2001)
- [24] Y. Ohkuwa and Y. Ezawa, Class. Quant. Grav. 29, 215004 (2012)
- [25] Y. Ohkuwa and Y. Ezawa, Class. Quant. Grav. 30, 235015 (2013)
- [26] Y. Ohkuwa, Int. J. Mod. Phys. A 13, 4091 (1998)
- [27] A. Buonanno, M. Gasperini, M. Maggiore and C. Ungarelli, Class. Quant. Grav. 14, L97 (1997)
- [28] N. Kontoleon and D. L. Wiltshire, Phys. Rev. D 59, 063513 (1999)
- [29] D. L. Wiltshire, Gen. Rel. Grav. 32, 515 (2000)
- [30] J. P. Luminet, J. Weeks, A. Riazuelo, R. Lehoucq and J. P. Uzan, Nature 425, 593 (2003)
- [31] M. L. Rey and J. P. Luminet, Phys. Rept. 254, 135 (1995)
- [32] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, Dover Publishing, New York (1972)
- [33] M. Maziashvili, Phys. Rev. D 71, 027505 (2005)
- [34] M. Faizal, Int. J. Geom. Meth. Mod. Phys. 12, 1550050 (2015)
- [35] G. Calcagni, S. Gielen and D. Oriti, Class. Quantum Grav. 29, 105005 (2012)
- [36] S. Gielen, JHEP 1508, 010 (2015)
- [37] G. Calcagni, Phys. Rev. D 90, 064047 (2014)
- [38] S. Gielen, D. Oriti and L. Sindoni, Phys. Rev. Lett. 111, 031301 (2013)
- [39] S. Upadhyay, Mod. Phys. Lett. A 30, 1550072 (2015)
- [40] S. Upadhyay, Gen. Rel. Grav. 46, 1678 (2014)
- [41] S. Upadhyay, Phys. Lett. B 740, 341 (2015)
- [42] S. Upadhyay, Ann. Phys. 356, 299 (2015)
- [43] S. Upadhyay, Phys. Lett. B 727, 293 (2013)
- [44] S. Upadhyay, Ann. Phys. 344, 290 (2014)
- [45] S. Upadhyay, Ann. Phys. 340, 110 (2014)
- [46] R. Banerjee, B. Paul and S. Upadhyay, Phys. Rev. D 88, 065019 (2013)
- [47] P. Horava, JHEP, 0903, 020 (2009)
- [48] P. Horava, Phys. Rev. Lett. 102, 161301 (2009)
- [49] P. Horava, Phys. Rev. D 79, 084008 (2009)

- [50] B. Vakili and V. Kord, *Gen. Rel. Grav.* 45, 1313 (2013)
- [51] M. Sakamoto, *Phys. Rev. D* 79, 124038 (2009)
- [52] R. Garattini, *Phys. Rev. D* 86, 123507 (2012)
- [53] R. Garattini and E. N. Saridakis, *Eur. Phys. J. C* 75, 343 (2015)
- [54] A. F. Ali, M. Faizal and M. M. Khalil, *JHEP* 1412, 159 (2014)
- [55] A. F. Ali, M. Faizal and M. M. Khalil, *Phys. Lett. B* 743, 295 (2015)
- [56] A. F. Ali, *Phys. Rev. D* 89, 104040 (2014)
- [57] A. F. Ali, M. Faizal and M. M. Khalil, *Nucl. Phys. B* 894, 341 (2015)
- [58] S. H. Hendi and M. Faizal, *Phys. Rev. D* 92, 044027 (2015)
- [59] A. F. Ali, M. Faizal and B. Majumder, *Europhys. Lett.* 109, 20001 (2015)
- [60] R. Garattini and B. Majumder, *Nucl. Phys. B* 883, 598 (2014)
- [61] R. Garattini and G. Mandanici, *Phys. Rev. D* 83, 084021 (2011)
- [62] M. Faizal, *Int. J. Mod. Phys. A* 29, 1450106 (2014)
- [63] M. Kober, *Int. J. Mod. Phys. A* 27, 1250106 (2012)
- [64] B. Vakili and H. R. Sepangi, *Phys. Lett. B* 651, 79 (2007)
- [65] P. Pedram, *Phys. Rev. D* 91, 063517 (2015)
- [66] A. Kempf, G. Mangano, and R. B. Mann, *Phys. Rev. D* 52, 1108 (1995)
- [67] M. Kober, *Phys. Rev. D* 82, 085017 (2010)
- [68] M. Faizal, *Int. J. Mod. Phys. A* 30, 1550036 (2015)